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A Generalization of Lagrange Interpolation Theorem*

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1. INTRODUCTION

Let f be an analytic function on the unit closed disk $\bar{D} = \{z \in \mathbb{C}: |z| \leqslant 1\}$. In notation $f \in A(\bar{D})$. Let $z_1, z_2, \ldots,$ be a sequence of points in $\bar{D}$ satisfying

$$z_n \text{ are all distinct and } \lim z_n = 0. \tag{1.1}$$

Define

$$P_{n-1}(z; f; z_1, z_2, \ldots, z_n) = \sum_{k=1}^{n} f(z_k) \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} (z - z_i) \right) \Bigg/ \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} (z_k - z_i) \right) \tag{1.2}$$

to be the Lagrange interpolation polynomial of f at $z_1, z_2, \ldots, z_n$. Then it is well known [3] that

$$\lim_{n} \|P_{n-1} - f\|_{\infty} = 0. \tag{1.3}$$

Here and in the following $\|\cdot\|_{\infty}$ will always be the sup norm on $\bar{D}$.

From (1.3) it follows that, given $\{f(z_k)\}$, we can recover $f(z)$ as a limit. It is natural to study a similar setting where, however, errors are allowed. More precisely, can we reconstruct f if we are given sequentially the data

$$f(z_1) + e_1,\ f(z_2) + e_2,\ f(z_3) + e_3, \ldots, \tag{1.4}$$

where the errors e_k are assumed to be independent and identically distributed random vectors on a probability space with mean (0, 0) and finite variance.

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By (1.3) it is natural to take

$$\sum_{k=1}^{n} (f(z_k)+e_k)\left(\prod_{\substack{i=1\\ i\neq k}}^{n} (z-z_i)\right)\Big/\left(\prod_{\substack{i=1\\ i\neq k}}^{n} (z_k-z_i)\right) \tag{1.5}$$

as our estimator. Unfortunately, this approach does not work [1]. This leads to the consideration of the following problem

$$\min_{g\in S_m} \sum_{k=1}^{n} |g(z_k)-(f(z_k)+e_k)|^2, \tag{1.6}$$

where $m \leqslant n-1$ and S_m is the set of all polynomials with order $\leqslant m$. Note that (1.5) is the solution to problem (1.6) if $m=n-1$. The reason why (1.5) is not suitable as an estimator to f is because the corresponding minimization space S_{n-1} is too big. According to Grenander's sieve method [5, 6], this difficulty can often be overcome if the minimum in (1.6) is taken over a sequence of smaller subspaces S_m, $m \leqslant n-1$, which grows slowly to $A(\bar{D})$ as $n \to \infty$.

Problem (1.6) can be regarded as a minimization problem over $2(m+1)$ real variables. By using elementary calculus it is not hard to show that (1.6) has a unique solution

$$(F_n+E_n)\cdot(A_{m+1,n})^*\cdot C_{m+1}^{-1}\cdot W_{m+1}, \tag{1.7}$$

where $F_n=(f(z_1), f(z_2),..., f(z_n))$, $E_n=(e_1, e_2,..., e_n)$, W_{m+1} is the $(m+1)\times 1$ column vector $(1, z,..., z^m)$,

$$A_{m+1,n}=(z_j^{i-1})_{(m+1)\times n}$$

is the $(m+1)\times n$ matrix with $(z_j)^{i-1}$ as its (i, j)th element, and

$$C_{m+1}=(A_{m+1,n})\cdot(A_{m+1,n})^*.$$

Here * means complex transposition. Note that C_{m+1}^{-1} exists, because C_{m+1} is positive definite by (1.1) and the assumption that $m \leqslant n-1$. By letting all $e_n=0$, it is clear that the deterministic part

$$F_{n,m}(z)=F_n\cdot(A_{m+1,n})^*\cdot C_{m+1}^{-1}\cdot W_{m+1} \tag{1.8}$$

of (1.7) is the solution to the problem

$$\min_{g\in S_m} \sum_{k=1}^{n} |g(z_k)-f(z_k)|^2, \qquad m\leqslant n-1 \tag{1.9}$$

and $F_{n,m}(z)$ should converge to $f(z)$ in some way if the sieve method is applicable to the estimation problem (1.4).

The main purpose of this paper is to verify this claim. In fact, $F_{n,m}(z)$ has the following representation.

THEOREM 1. *Let $z_1, z_2, \ldots, z_n$ be n distinct points and f any function. Then the unique solution $F_{n,m}(z)$ to problem* (1.9) *satisfies*

$$F_{n,m}(z) = \frac{\sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n} \left(\prod_{1 \leqslant j < k \leqslant m+1} |z_{i_j} - z_{i_k}|^2\right) P_m(z; f; z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}})}{\sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n} \left(\prod_{1 \leqslant j < k \leqslant m+1} |z_{i_j} - z_{i_k}|^2\right)}, \tag{1.10}$$

where $P_m(z; f; z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}})$, as in (1.2), *is the* mth-*order Lagrange interpolation polynomial of f at $z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}}$.*

In view of (1.3) it is not hard to believe the following result.

THEOREM 2. *Let $f \in A(\bar{D})$ and $\{z_n\}$ satisfy* (1.1). *Then*

$$\lim \|F_{n,m}(z) - f(z)\|_\infty = 0$$

as both n, m tend to ∞ with $m \leqslant n-1$.

Finally, let us say a few words on the estimation problem (1.4). In order that there is a restoration algorithm working for every $f \in A(\bar{D})$, it is necessary and sufficient that $\{z_n\}$ has infinite convergence exponent, i.e.,

$$\sum_{n=1}^{\infty} |z_n|^p = \infty \qquad \text{for every} \quad p > 0.$$

This result will appear in [2]. There, how $m = m(n)$ tends to ∞ will depend on $\{z_1, z_2, \ldots, z_n\}$ and is not explicitly known.

2. PROOF OF THEOREM 1

Define the symmetric sums $T_k(z_1, z_2, \ldots, z_u)$ as follows

$$\prod_{i=1}^{u} (z - z_i) = \sum_{k=0}^{u} (-1)^k T_k(z_1, z_2, \ldots, z_u) z^{u-k}. \tag{2.1}$$

We shall first prove a lemma on C_{m+1}.

LEMMA. *Let $C_{m+1}^{-1} = (d(i, j))_{(m+1) \times (m+1)}$. Then*

$$\det C_{m+1} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n} \left(\prod_{1 \leqslant j < k \leqslant m+1} |z_{i_j} - z_{i_k}|^2 \right) \tag{2.2}$$

and for $0 \leqslant r,\ s \leqslant m$,

$$d(r+1, s+1) = (-1)^{r+s} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} T_{m-s}(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) \times T_{m-r}(\bar{z}_{i_1}, \bar{z}_{i_2}, \ldots, \bar{z}_{i_m}) \times \left(\prod_{1 \leqslant j < k \leqslant m} |z_{i_j} - z_{i_k}|^2 \right) \Big/ (\det C_{m+1}). \tag{2.3}$$

Proof. Since $C_{m+1} = (A_{m+1,n}) \cdot (A_{m+1,n})^*$, the Binet–Cauchy formula [4] tells us that $\det C_{m+1}$ and $d(r+1, s+1)$ can be expressed as sums of products of the corresponding minors of $A_{m+1,n}$ and $(A_{m+1,n})^*$. Equalities (2.2) and (2.3) will then be proved if one notes that

$$\det A_{n,n} = \prod_{1 \leqslant j < i \leqslant n} (z_i - z_j)$$

and the $n \times n$ matrix obtained from $A_{n+1,n}$ by deleting its $(k+1)$th row $(z_1^k, z_2^k, \ldots, z_n^k)$ has determinant

$$\prod_{1 \leqslant j < i \leqslant n} (z_i - z_j) \cdot T_{n-k}(z_1, z_2, \ldots, z_n). \qquad \text{Q.E.D.}$$

Now we may prove Theorem 1. By symmetry it is enough to show that for each $0 \leqslant s \leqslant m$, $f(z_1)\, z^s$ has the same coefficient on both sides of (1.10). By (1.2), (2.1), and (2.2), this is equivalent to show that

$$\sum_{r=0}^{m} \bar{z}_1^r d(r+1, s+1) = \sum_{2 \leqslant i_2 < i_3 < \cdots < i_{m+1} \leqslant n} \left(\prod_{k=2}^{m+1} (\bar{z}_1 - \bar{z}_{i_k}) \right) \left(\prod_{2 \leqslant j < k \leqslant m+1} |z_{i_j} - z_{i_k}|^2 \right) \times (-1)^{m-s} T_{m-s}(z_{i_2}, \ldots, z_{i_{m+1}}) / (\det C_{m+1}). \tag{2.4}$$

Here we have used the fact that $|z_1 - z_i|^2 = (z_1 - z_i)(\bar{z}_1 - \bar{z}_i)$. By (2.1) and (2.3), the left-hand side of (2.4) equals

$$(-1)^{m+s} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} T_{m-s}(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) \cdot \left(\prod_{k=1}^{m} (\bar{z}_1 - \bar{z}_{i_k}) \right) \times \left(\prod_{1 \leqslant j < k \leqslant m} |z_{i_j} - z_{i_k}|^2 \right) \Big/ (\det C_{m+1}).$$

In order that $\prod_{k=1}^{m} (\bar{z}_1 - \bar{z}_{i_k}) \neq 0$ it is necessary that $i_1 \geqslant 2$. Now change the index from i_k to i_{k+1}, we get the right-hand side of (2.4). This completes the proof.

3. PROOF OF THEOREM 2

Since $f \in A(\bar{D})$, there is a number $r > 1$ such that f is analytic on $\{z: |z| \leqslant r\}$. Under the assumption (1.1), it is well known [3, Theorem 3.6.1] that $f(z) - P_m(z; f; z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}})$ equals the following contour integral

$$(2\pi i)^{-1} \int_C \left[f(t) \prod_{k=1}^{m+1} (z - z_{i_k}) \right] \Big/ \left[(t-z) \prod_{k=1}^{m+1} (t - z_{i_k}) \right] dt,$$

where $C = \{z: |z| = r\}$. Because $\lim z_n = 0$, it can be shown easily [3, Theorem 4.4.3] that there are positive constants M, δ with $r - \delta > 1$, such that

$$\| f(z) - P_m(z; f; z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}}) \|_\infty \leqslant M(r - \delta)^{-m}$$

holds for all $\{z_{i_1}, z_{i_2}, \ldots, z_{i_{m+1}}\}$. Then the theorem follows from Theorem 1.

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REFERENCES

1. Y-S. CHOW, "Estimation of Conformal Mappings," Ph.D. thesis, Brown Univ., Providence, R.I., 1980.
2. Y-S. CHOW, Estimation of analytical functions, submitted for publication.
3. P. J. DAVIS, "Interpolation and Approximation," Dover, New York, 1975.
4. F. R. GANTMACHER, "The Theory of Matrices," Vol. 1, Chelsea, New York, 1957.
5. S. GEMAN AND C-R. HWANG, Nonparametric likelihood estimation by the method of sieves, *Ann. Statist.* **10** (1982), 401–414.
6. U. GRENANDER, "Abstract Inference," Wiley, New York, 1981.